A Study on Linear Process Control

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Critical limits of controller parameters (PID) and critical cycling periods are derived for several linear processes. A method for determining the approximate dynamic characteristics of a process is developed. A pressure control system is analyzed by this method and its behavior interpreted.

Of the many studies published on process control some are highly mathematical with limited usefulness. Convenient tables, relating the range of controller parameters (PID), and critical cycling periods are herein presented for typical simple linear processes.

Though there are several methods for estimating process dynamic characteristics by means of various responses, the step-response method does not have good accuracy in general, and the frequency response is rather involved. Likewise other methods have disadvantages. A method is described in which the approximate dynamic characteristics of a process are obtained by use of critical cycling tests, and the control behavior of a simple process is interpreted. This method is simple and practical and is considered as a supplement to those mentioned.

CRITICAL CONDITIONS OF CONTROLLER PARAMETERS

Several methods for determining the optimum controller settings have been reported since the publication of the Ziegler-Nichols method (*1, 2, 9*). These are usually based upon a 25% damping ratio. However in some processes the overshooting of the response curve is to be avoided, in which case a near critical damping response is desirable. On the other hand if the output amplitudes are not excessive, a nearly critical cycling condition may be selected if a quick response is important. Thus the requirements of the process dictate the type of the optimum response selected. Moreover the actual controller performance is not strictly in agreement with the theoretical control response, but under all circumstances the optimum control should be selected between conditions of critical cycling and critical damping. The conditions for some typical linear processes are here presented. This treatment is based on the well-known mathematical relations of automatic control (*6, 7*).

A third-order process and PID control represents the highest order system which can be easily analyzed:

Process:

$$G = \frac{M}{(T_1 s + 1)(T_2 s + 1)(T_3 s + 1)} \quad (1)$$

Controller:

$$H = K\left(1 + \frac{1}{T_I s} + T_D s\right) \quad (2)$$

The characteristic equation of this closed system is obtained from $GH + 1 = 0$ as follows:

$$T_1T_2T_3T_Is^4 + (T_1T_2 + T_2T_3 + T_3T_1)T_Is^3 + (T_1 + T_2 + T_3 + MKT_D)T_Is^2 + (MK+1)T_Is + MK = 0 \quad (3)$$

The stable conditions of the closed loop are obtained from the Routh-Hurwitz theorem, and the damped conditions result from the existence of imaginary parts in the roots of the characteristic equation, as shown in Table 1.

The critical (undamped) cycling period of these systems can be obtained by equating the imaginary portion of

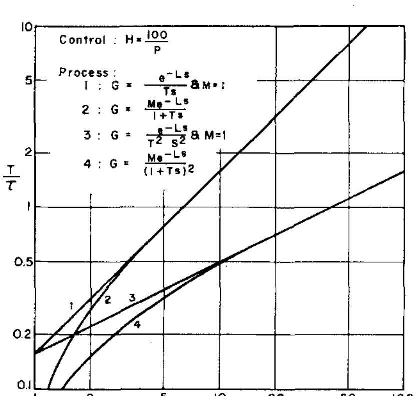

Fig. 1. The calculation of *T* from critical cycling in four processes with proportional control.

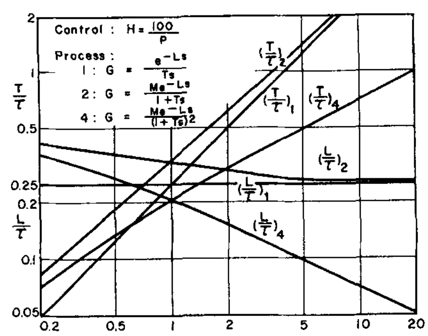

Fig. 2. Relations between *T*/τ and *L*/τ vs. *T*/*L* for three processes with proportional control and critical cycling.

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TABLE 1. CONTROL CONDITIONS FOR LINEAR PROCESS (P, I, D CONTROL)

PROCESS	CONTROL / RESPONSE	P	PI	PD	PID
	H =	K	$K\left(1+\frac{1}{T_I s}\right)$	$K(1+T_D s)$	$K\left(1+\frac{1}{T_I s}+T_D s\right)$
1st ORDER	STABLE	ALWAYS	ALWAYS	ALWAYS	ALWAYS
$G=\frac{M}{1+TS}$	DAMPED CYCLING	ALWAYS OVERDAMPED	$T_I<\frac{4MKT}{(MK+1)^2}$	ALWAYS OVERDAMPED	$T_I<\frac{4MK(T+MKT_D)}{(MK+1)^2}$
2nd ORDER	STABLE	ALWAYS	$T_I>\frac{MK}{MK+1}\cdot\frac{T_1T_2}{T_1+T_2}$	ALWAYS	$T_I>\frac{MK}{MK+1}\cdot\frac{T_1T_2}{T_1+T_2+MKT_D}$
$G=\frac{M}{(1+T_1S)(1+T_2S)}$	DAMPED CYCLING	$MK>\frac{(T_1-T_2)^2}{4T_1T_2}$	———	$T_D<\frac{1}{MK}\left\{2\sqrt{T_1T_2(MK+1)}-(T_1+T_2)\right\}$	———
3rd ORDER $G=\frac{M}{(1+T_1s)(1+T_2s)(1+T_3s)}$	STABLE	$MK<(T_1+T_2+T_3)\times\left(\frac{1}{T_1}+\frac{1}{T_2}+\frac{1}{T_3}\right)-1$	$T_I>\frac{MK}{MK+1}\times\frac{T_1T_2T_3\left(\frac{1}{T_1}+\frac{1}{T_2}+\frac{1}{T_3}\right)^2}{\left(\frac{1}{T_1}+\frac{1}{T_2}+\frac{1}{T_3}\right)(T_1+T_2+T_3)-(MK+1)}$	$T_D>\frac{1}{MK}\left\{\frac{MK+1}{\frac{1}{T_1}+\frac{1}{T_2}+\frac{1}{T_3}}-(T_1+T_2+T_3)\right\}$	$T_D>\frac{1}{MK}\left\{\frac{MK+1}{\frac{1}{T_1}+\frac{1}{T_2}+\frac{1}{T_3}}-(T_1+T_2+T_3)\right\}$ $T_I>\frac{MK}{MK+1}\cdot\frac{T_1T_2T_3\left(\frac{1}{T_1}+\frac{1}{T_2}+\frac{1}{T_3}\right)^2}{\left(\frac{1}{T_1}+\frac{1}{T_2}+\frac{1}{T_3}\right)(T_1+T_2+T_3+MKT_D)-(MK+1)}$
2nd ORDER	STABLE	ALWAYS	$T_I>\frac{MK}{MK+1}\cdot\frac{T}{2}$	ALWAYS	$T_I>\frac{MK}{MK+1}\cdot\frac{T^2}{2T+MKT_D}$
$G=\frac{M}{(1+Ts)^2}$	DAMPED CYCLING	ALWAYS	———	$T_D<\frac{2T}{MK}(\sqrt{MK+1}-1)$	———
3rd ORDER $G=\frac{M}{(1+Ts)^3}$	STABLE	$MK<8$	$T_I>\frac{9MKT}{(MK+1)(8-MK)}$	$T_D>\frac{T}{3}\left(1-\frac{8}{MK}\right)$	$T_D>\frac{T}{3}\left(1-\frac{8}{MK}\right)$ $T_I>\frac{9MKT^2}{(MK+1)\{(8-MK)T+3MKT_D\}}$

the equation to zero, when letting $s=j\omega$ and $\tau=2\pi/\omega$. From equation (3) of a third-order process

$$\tau=2\pi\sqrt{\frac{T_1T_2+T_2T_3+T_3T_1}{MK+1}} \quad (4)$$

and for a second-order process

$$\tau=2\pi\sqrt{\frac{T_1T_2}{1+MK}} \quad (5)$$

From (4) and (5) it is observed that the critical cycling period is independent of T_I and T_D of a controller. It is noteworthy that this result can be extended to any higher-order process, of which the transfer function is the type of Equation (1).

ESTIMATION TRANSFER FUNCTION BY CRITICAL CYCLING METHOD (5)

The critical (undamped) cycling method has been used by Ziegler and Nichols in order to determine the optimum control conditions (9). It can also give dynamic properties of a process by making use of critical conditions and the critical cycling period for the closed-loop system. In the Ziegler-Nichols method a first-order and dead-time transfer function is usually assumed to approximate the process characteristics. However this function may be unsuitable to approximate the actual characteristics because of its discontinuity at the end of the dead time. Therefore a second-order transfer function with two equal time constants and a dead time is presented to approximate any usual process characteristics as follows:

$$G=\frac{M\,e^{-Ls}}{(Ts+1)^2} \quad (6)$$

If this process is controlled by a proportional action ($H=100/P$) and the closed loop is in the critical cycling, the following equations are obtained:

$$GH=\frac{100\,M\,e^{-Ls}}{P(Ts+1)^2}=-1 \quad (7)$$

which in substituting $s=j\omega$ and separating into real and imaginary parts yields

$$T=\frac{\tau_p}{2\pi}\sqrt{\frac{100M}{P}-1} \quad (8)$$

$$L=\tau_p\left(\frac{1}{2}-\frac{1}{\pi}\tan^{-1}2\pi\frac{T}{\tau_p}\right)$$

Two unknown characteristics, T and L, can be easily calculated from known values, M, P, and τ_p which are obtained by a statical change and a critical cycling test. This method is con-

TABLE 2. THE CALCULATIONS OF T AND L FROM CRITICAL CYCLING IN SEVERAL PROCESSES WITH P OR PI CONDITIONS

PROCESS G	CONTROL H	TIME CONSTANT T	DEAD TIME L
$\frac{Me^{-Ls}}{(1+Ts)^n}$	$\frac{100}{P}$	$\frac{\tau}{2\pi}\sqrt{\left(\frac{100M}{P}\right)^{2/n}-1}$	$\frac{\tau}{2}\left(1-\frac{n}{\pi}\tan^{-1}2\pi\frac{T}{\tau}\right)$
$\frac{Me^{-Ls}}{(1+T_1s)(1+T_2s)}$	$\frac{100}{P}$	$\left\{1+\left(2\pi\frac{T_1}{\tau}\right)^2\right\}\left\{1+\left(2\pi\frac{T_2}{\tau}\right)^2\right\}=\left(\frac{100M}{P}\right)^2$	$\frac{\tau}{2}\left(1-\frac{1}{\pi}\tan^{-1}2\pi\frac{T_1}{\tau}-\frac{1}{\pi}\tan^{-1}2\pi\frac{T_2}{\tau}\right)$
$\frac{e^{-Ls}}{Ts}$	$\frac{100}{P}\left(1+\frac{1}{T_Is}\right)$	$\frac{100\tau}{2\pi P}\sqrt{\left(\frac{\tau}{2\pi T_I}\right)^2+1}$	$\frac{\tau}{2\pi}\tan^{-1}2\pi\frac{T_I}{\tau}$
$\frac{Me^{-Ls}}{(1+Ts)^n}$	$\frac{100}{P}\left(1+\frac{1}{T_Is}\right)$	$\frac{\tau}{2\pi}\sqrt{\left(\frac{100M}{P}\right)^{2/n}\left\{\left(\frac{\tau}{2\pi T_I}\right)^2+1\right\}^{1/n}-1}$	$\frac{\tau}{2}\left(\frac{1}{2}+\frac{1}{\pi}\tan^{-1}2\pi\frac{T_I}{\tau}-\frac{n}{\pi}\tan^{-1}2\pi\frac{T}{\tau}\right)$
$\frac{Me^{-Ls}}{(1+T_1s)(1+T_2s)}$	$\frac{100}{P}\left(1+\frac{1}{T_Is}\right)$	$\left\{1+\left(2\pi\frac{T_1}{\tau}\right)^2\right\}\left\{1+\left(2\pi\frac{T_2}{\tau}\right)^2\right\}=\left(\frac{100M}{P}\right)^2\left\{\left(\frac{\tau}{2\pi T_I}\right)^2+1\right\}$	$\frac{\tau}{2}\left(\frac{1}{2}+\frac{1}{\pi}\tan^{-1}2\pi\frac{T_I}{\tau}-\frac{1}{\pi}\tan^{-1}2\pi\frac{T_1}{\tau}-\frac{1}{\pi}\tan^{-1}2\pi\frac{T_2}{\tau}\right)$

TABLE 3. EXPERIMENTAL AND CALCULATED DYNAMIC CHARACTERISTICS

Tank volume, cu. ft.	Tank pressure, psi lb./sq. in. or %	Critical cycling, P %	sec.	Case A, L sec.	T sec.	Case B, L' sec.	T' sec.	Case C, Lo sec.	Tl sec.	Step response, L" sec.	T" sec.
1.8	20	11	30	2.0	22	8	108	2.0	25		
1.8	30	13	25	1.8	17	6	76	1.9	14		
1.8	40	14	22	1.7	14	6	63	1.8	10	~10	~30
1.8	50	15	19	1.5	12	5	50	1.7	7		
1.8	60	13	18	1.3	12	5	55	1.5	7		
5.5	20	10	53	3.4	41	14	210	4.2	82	~15	~50
9.1	20	8	60	3.5	53	15	298	5.1	128		
										~20	~70
9.1	30	9	46	2.8	38	12	204	3.3	70		

venient to obtain dynamic characteristics of a process without any special measuring instrument, especially in a process having a quick response. This procedure may also be applied to other control actions, for example PI control. If the critical cycling test were performed with two different control actions, four unknown characteristics might be evaluated in the closed loop.

Another advantage of this method is that a cycling period τ can be measured very easily and accurately only by the use of a stopwatch. But the proportional band (or sensitivity) of the controller has to be calibrated. Any nonlinearity in the closed loop except dead time should be avoided during the test or by a computation procedure.

Figures 1 and 2 show relations among the above mentioned characteristic values and are useful to calculate T and L. These graphs also include the cases of simpler processes which often appear in practice. Table 2 gives the relationships in more general processes controlled by P or PI action.

If hysteresis is present in the controller, that is backlash, the true T' and L' are smaller than the T and L values calculated by this method without the hysteresis. The difference between the values depends upon the amplitude of the critical cycling. These values are obtained by using the describing function shown in Figure 3 as follows (4, 8):

$$T' = T\, x\, |G'| \text{ and } L' = L \times \frac{90 - \angle G'}{360} \quad (9)$$

A PRESSURE-CONTROL EXPERIMENT

Pressure of a tank is controlled by proportional action with two manipulating valves, as shown in Figure 4. The process may be assumed as follows:

$$C\frac{dp}{dt} = C\frac{d\Delta p}{dt} = \Delta q_1 - \Delta q_2 \quad \text{[lb./sec.]}$$

$$\Delta q_1 = \frac{-\Delta p}{R_1} + K_{v1}\,\Delta y \quad \text{[lb./sec.]} \quad (10)$$

$$\Delta q_2 = \frac{\Delta p}{R_2} - K_{v2}\,\Delta y \quad \text{[lb./sec.]}$$

The flow-rate increments q_1 and q_2 can be eliminated in the above equations:

$$C\frac{d\Delta p}{dt} + \frac{\Delta p}{R} = K\ \Delta_y \quad (11)$$

where $1/R = 1/R_1 + 1/R_2$ and $K_v = K_{v1} + K_{v2}$.

The transfer function of the process is obtained as follows:

$$G = \frac{\Delta p(s)}{k_2\,\Delta y(s)} = \frac{K_v/k_2}{Cs + \frac{1}{R}} = \frac{M}{T_1 s + 1} \quad (12)$$

where $M = RK_v/k_2$ and $T_1 = CR$.

The transfer function of the controller and its associated element may be approximated to include a first-order delay and a dead time, which in this case

$$H = \frac{\Delta y(s)}{\Delta p(s)} = \frac{100}{P}\frac{e^{-L_o s}}{(T_2 s + 1)} \quad (13)$$

and the following open loop transfer function becomes

$$GH = \frac{100\,M\,e^{-L_o s}}{P(T_1 s + 1)(T_2 s + 1)} \quad (14)$$

where $M \doteq 2.5$ was determined by static experiments. This represents an average value, since the magnitude of M changes with the tank pressure. By a separate determination T_2 is approximately 20 sec.

The approximate transfer function by the critical cycling method is assumed to be one of the following:

Case A: Equation (6)

Case B: $$GH = \frac{100\,M\,e^{-L's}}{P(1 + T's)} \quad (15)$$

Case C: Equation (14)

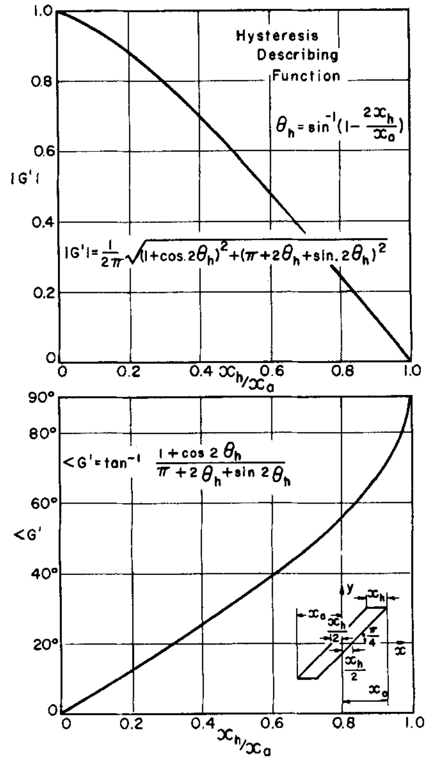

Fig. 3. Describing function chart of hysteresis.

The characteristic values calculated from Equation (8) and Table 2 are reported in Table 3. In this system Case C corresponds to the correct dynamic characteristic values, and Case A gives more reasonable values than Case B. A chart record of the critical cycling is shown in Figure 5.

When one assumes Case B, the approximate experimental values, T'' and L'', obtained by the step response test are also reported in Table 3. Since the actual response is similar to a second order, it is difficult to obtain accurate values by the step method. In all cases the time constant T of the process should be proportional to the tank capacity C or the tank volume V for a given tank pressure. The dead time L should be very small. Case C yields such results. As shown in Table 3 when the tank volume is increased, the closed loop system becomes more stable. Therefore the proportional band decreases and the cycling period increases at the critical cycling.

INSTABILITY OF PRESSURE CONTROL BY TWO VALVES (3)

From the experimental results, which are given in Table 3 and Figure 5, it is evident that at the intermediate

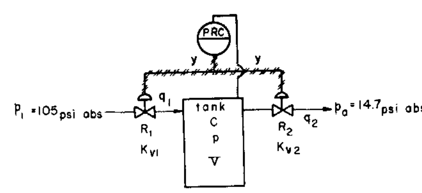

Fig. 4. Pressure control of a tank.

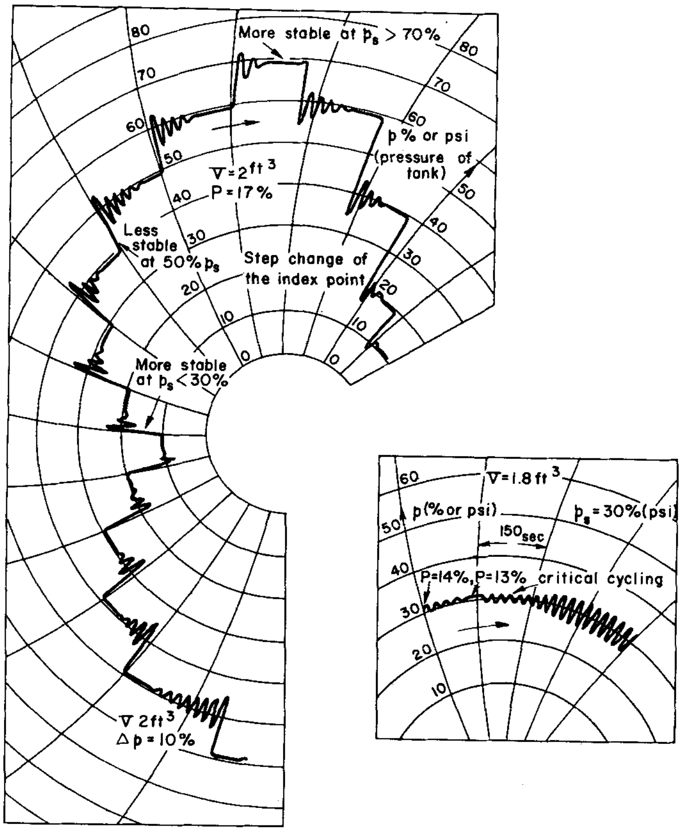

Fig. 5. Example of control of air pressure in the tank by a simple proportional control action.

pressure ($p \doteq 50$ lb./sq. in. or %) the system becomes less stable than that at higher and lower pressures with the same control conditions. This phenomenon may be explained as follows:

1. The pressure above the inlet control valve is about $p_1 = 90$ lb./sq. in. (90%), and the discharge pressure at the outlet control valve is atmospheric, $p_a = 0$ lb./sq. in. (0%). The pipeline has some resistance from a silencer. Therefore if the tank pressure p is about 50 lb./sq. in. (50%), the flow through the inlet and the outlet valves become sonic. With large values of R Equation (12) approaches an integral characteristic of the form $G = K_v/C\,k_2 s$, and the process becomes less stable at the intermediate pressure than at the higher and lower pressures.

2. In general if pressure drop across a control valve increases but below sonic condition, the value of K_{v1} in Equation (10) increases and the static process gain M also increases. Then the process becomes less stable than at a smaller pressure drop across the valve. When the tank pressure is low, the flow is sonic through the inlet valve and subsonic through the outlet valve. With a high tank pressure the flow conditions are reversed at intermediate tank pressure, the pressure drop increases across that valve in which the flow is subsonic, and the process becomes less stable.

CONCLUSION

The critical relations between controller parameters (PID) for several typical linear processes are developed and represented in Table 1 which is useful for actual operation of a process controller. By mathematical analysis it was demonstrated that the critical cycling periods of these systems are independent of the integral and derivative actions of the controller.

A method, requiring only simple time measuring instruments, is described for estimating the approximate dynamic characteristics of some common processes within a closed-loop system. The method is also applicable to processes whose characteristic values are too small to be measured by a step change.

A second-order process with two equal time constants and a dead time is recommended over a first-order process as an approximate transfer function of a process.

The dynamic characteristics of a pressure-control system were obtained by the method described and different pressures and tank volumes. The variation in stability of this system was accounted for by a change in transfer function of the process in the sonic and subsonic flows.

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NOTATION

- a = coefficient
- C = capacity, sq. ft.
- G = transfer function of a process (-)
- H = transfer function of a controller (-)
- K = proportional sensitivity = $100/P$ (-)
- K_v = valve coefficient, lb./(sec.) · (% of manipulating signal)
- k = constant
- k_2 = lb./(sq. ft. x % of manipulating signal)
- L = dead time, min. or sec.
- M = process static gain (-) = $\left[\dfrac{\%\text{ of chart scale}}{\%\text{ of manipulating signal}}\right]$
- P = proportional band $\left[\dfrac{\%\text{ of chart scale}}{100\%\text{ of manipulating signal}}\right]$
- p = pressure, controlled variable, lb./sq. ft.
- q = flow rate, lb./sec.
- R = resistance coefficient of a valve, sec./sq. ft.
- s = Laplace parameter, 1/min. or 1/sec.
- T = time constant of a process, min. or sec.
- T_I = integral time of a controller, min. or sec.
- T_D = derivative time of a controller, min. or sec.
- V = total volume of a tank, cu. ft.
- x = amplitude
- y = manipulating signal, %
- τ = period of cycling, min. or sec.
- ω = angular frequency, rad./min. or rad./sec.

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